

# Infrared Renormalization Group Flow for Heavy Quark Masses

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A short-distance heavy quark mass depends on two parameters, the renormalization scale  $\mu$  controlling the absorption of ultraviolet fluctuations into the mass, and a scale  $R$  controlling the absorption of infrared fluctuations.  $1/R$  can be thought of as the radius for perturbative corrections that build up the mass beyond its point-like definition in the pole scheme. Treating  $R$  as a variable gives a renormalization group equation. We argue that the sign of this anomalous dimension is universal: increasing  $R$  to add IR modes decreases  $m(R)$ . The flow improves the stability of conversions between mass schemes, allowing us to avoid large logs and the renormalon. The flow in  $R$  can be used to study IR renormalons without using bubble chains, and we use it to determine the coefficient of the  $\mathcal{O}(\Lambda_{\text{QCD}})$  renormalon ambiguity of the pole mass with a convergent sum-rule.

The pole-mass,  $m_{\text{pole}}$ , provides a simple definition of a mass-parameter in perturbative quantum field theory, corresponding to the location of the single particle pole in the two-point function. For the electron mass in QED  $m_{\text{pole}}$  is used almost exclusively, but for quarks in QCD there are two reasons it is impractical. First, at high energies, large logs appear which spoil perturbation theory with  $m_{\text{pole}}$ . This problem is cured by introducing the concept of a running-mass  $m(\mu)$ , where the renormalization group (RG) flow in  $\mu$  is controlled by a mass-anomalous dimension. The second, and more serious problem, is that due to confinement there is no pole in the quark-propagator in non-perturbative QCD. Thus the concept of a quark pole-mass is ambiguous by  $\Delta m_{\text{pole}} \sim \Lambda_{\text{QCD}}$ . This ambiguity appears as a linear sensitivity to infrared momenta in Feynman diagrams, and results in a diverging perturbation series for any observable expressed in terms of  $m_{\text{pole}}$ , with terms  $\sim 2^n n! \alpha_s^{n+1}$  asymptotically for large  $n$ . For the heavy quark masses (charm, bottom, top) that we study, this behavior is referred to as the pole-mass  $\mathcal{O}(\Lambda_{\text{QCD}})$  renormalon problem [1], where the Borel transform of the series has a singularity at  $u = 1/2$ . Schemes without this infrared problem are known as short-distance masses, and always depend on an additional infrared scale  $R$ .

Typically,  $R$  is considered as intrinsic to the short-distance quark mass definition,  $m_R(\mu)$ . Examples are

$$\begin{aligned} \overline{\text{MS}} : & \quad \overline{m}(\mu), & R = \overline{m}(\mu); \\ \text{RGI [2]} : & \quad m_{\text{RGI}}, & R = m_{\text{RGI}}; \\ \text{kinetic [3]} : & \quad m_{\text{kin}}, & R = \mu_f^{\text{kin}}; \\ \text{1S [4]} : & \quad m_{1\text{S}}, & R = m_{1\text{S}} C_F \alpha_s(\mu); \\ \text{PS [5]} : & \quad m_{\text{PS}}, & R = \mu_f^{\text{PS}}. \end{aligned} \quad (1)$$

where  $C_F = 4/3$ . Many schemes have  $R = m$ , but this is not generic. For instance, the 1S-mass is defined as one-half the mass of the heavy quarkonium  $^3S_1$  state in perturbation theory, and its  $R$  is of order the inverse Bohr radius. In the kinetic and the potential subtraction (PS) schemes  $R$  is set by cutoffs,  $\mu_f^{\text{kin}}$  and  $\mu_f^{\text{PS}}$ , on integrals

over a heavy-quark correlator and the heavy-quark static potential respectively. Depending on the scales involved in a process, schemes with a specific range of  $\mu$  and  $R$  are most appropriate to achieve stable perturbative results.

The goal of this letter is to consider  $R$  as a continuous parameter, and study the RG flow in  $R$  of masses  $m(R, \mu) = m_R(\mu)$ . We consider converting between mass schemes  $m_A(R, \mu)$  and  $m_B(R', \mu)$  where  $R \ll R'$ . To avoid the  $\mathcal{O}(\Lambda_{\text{QCD}})$  renormalon in fixed-order perturbation theory a common expansion in  $\alpha_s(\mu)$  must be used, which inevitably introduces large logs,  $\ln(R'/R)$ . The RGE in  $R$  allows mass-scheme conversions to be done avoiding both large logs and the renormalon. We show this improves the stability of conversions between the  $\overline{\text{MS}}$  scheme with  $R = m$ , and low energy schemes with  $R \ll m$  that are extensively used for high precision determinations of heavy quark masses [6]. The solution of this RGE is also used to systematically derive a convergent series for the normalization of the  $u = 1/2$  singularity in the pole-mass Borel transform.

To start, translate the bare-quark mass in QCD to the pole-mass,  $m_{\text{bare}} = Z_m m_{\text{pole}}$ , where UV divergences from scales  $p^2 \gg m^2$  appear in the mass-renormalization constant  $Z_m$ . The difference between using  $m_{\text{pole}}$  and any other scheme  $m(R, \mu)$  corresponds to specifying additional finite subtractions,  $\delta m(R, \mu)$ . Let

$$m_{\text{pole}} = m(R, \mu) + \delta m(R, \mu), \quad (2)$$

$$\delta m(R, \mu) = R \sum_{n=1}^{\infty} \sum_{k=0}^n a_{nk} \left[ \frac{\alpha_s(\mu)}{4\pi} \right]^n \ln^k \left( \frac{\mu}{R} \right).$$

Here  $a_{nk}$  are numbers, and  $\alpha_s$  is in the  $\overline{\text{MS}}$ -scheme with

$$\frac{d\alpha_s(\mu)}{d \ln \mu} = \beta[\alpha_s(\mu)] = -2\alpha_s(\mu) \sum_{n=0}^{\infty} \beta_n \left[ \frac{\alpha_s(\mu)}{4\pi} \right]^{n+1}. \quad (3)$$

We will only consider gauge independent short-distance mass schemes for  $m(R, \mu)$ , where  $\delta m$  eliminates the infrared ambiguity associated to the pole mass. This requires that  $a_{(n+1)0} \sim 2^n n!$  asymptotically for large  $n$ .

These mass schemes come in two categories. In  $\mu$ -independent schemes (such as RGI, kinetic, 1S):  $\frac{d}{d \ln \mu} m(R, \mu) = 0$ . In these schemes  $a_{11} = 0$ , and  $a_{nk}$  with  $k \geq 1$  are determined by the  $a_n \equiv a_{n0}$ 's and  $\beta_n$ 's. Thus the  $a_n$ 's specify the scheme. Masses in the other category have a  $\mu$ -anomalous dimension (like  $\overline{\text{MS}}$ ), and using  $d/d \ln \mu m_{\text{pole}} = 0$  one finds  $\frac{d}{d \ln \mu} m(R, \mu) = -R\gamma_\mu[\alpha_s(\mu)]$ . Here  $a_{n1}$  and  $a_{n0}$  are needed to specify the scheme and  $\gamma_\mu$ .  $\gamma_\mu$  does not depend on  $\ln(\mu/R)$ , so all  $a_{nk}$  with  $k \geq 2$  are determined. For “mass-independent” schemes like  $\overline{\text{MS}}$  we always have  $a_{11} = 6C_F$ , and a universal  $\gamma_\mu$  at leading order (LO).

Eq. (2) can be used to identify  $R$  for schemes like those in Eq. (1). To see that  $R$  is related to absorbing IR fluctuations into the mass, consider the PS scheme where

$$m^{\text{PS}}(R) - m_{\text{pole}} = -\delta m^{\text{PS}}(R) \equiv \frac{1}{2} \int_{|q| < R} \frac{d^3 q}{(2\pi)^3} V(q). \quad (4)$$

Here  $V(q)$  is the momentum space color singlet static potential between infinitely heavy test charges in the  $\mathbf{3}$  and  $\bar{\mathbf{3}}$  representations. In  $m_{\text{pole}} - \delta m^{\text{PS}}$  the low momentum part of the potential precisely cancels the infrared sensitivity of  $m_{\text{pole}}$ , leaving a well-defined short-distance mass,  $m^{\text{PS}}$ . If we increase  $R$  from  $R_0$  to  $R_1$  then

$$-\delta m^{\text{PS}}(R_1) = \int_0^{R_0} dq \frac{q^2 V(q)}{4\pi^2} + \int_{R_0}^{R_1} dq \frac{q^2 V(q)}{4\pi^2}, \quad (5)$$

so additional potential energy is absorbed into  $m^{\text{PS}}$ , increasing the range of IR fluctuations included in the PS-mass. In other mass-schemes the precise definition of  $R$  differs, but the interpretation of this scale as an IR-cutoff still remains. Another simple example is what we call the static-scheme. The static energy,  $E^{\text{static}}(r) = 2m_{\text{pole}} + V(r)$  is free of the  $\mathcal{O}(\Lambda_{\text{QCD}})$  renormalon, which requires a cancellation of IR sensitivity between the pole-mass and the potential energy  $V(r)$ . Here  $V(r)$  is the Fourier transform of  $V(q)$ . A short-distance static mass,  $m^{\text{stat}}(R)$ , can be defined to make this cancellation explicit, with  $\delta m^{\text{stat}}(R) = -\frac{1}{2} V(r = 1/R)$ . [We use the modern definition of the static potential, which becomes  $\mu$ -dependent at  $\mathcal{O}(\alpha_s^4)$  [9], but does not suffer from infrared divergences. Since  $\mathcal{O}(\alpha_s^4)$  is beyond the order of our analysis we refer to the PS and static masses as  $\mu$ -independent. For convenience we also consider the kinetic scheme in the heavy-quark limit, so there are no higher powers of  $R$  in Eq. (2).] Since  $V$  is attractive, increasing  $R$  decreases the PS and the static mass. We will see that this decrease is universal. It is described by an RGE in  $R$  where  $d/d \ln R m(R) < 0$  at LO in  $\alpha_s(R)$  for all known physical mass schemes.

**The RGE for  $R$ .** Consider any  $\mu$ -independent scheme where  $R$  is a free parameter, such as the PS, static, and kinetic schemes. The pole-mass in Eq. (2) is  $R$ -independent, so  $Rd/dR m(R) = -Rd/dR \delta m(R)$ . To avoid having large  $\ln(\mu/R)$ 's on the RHS we must expand in  $\alpha_s(R)$ ,  $\delta m(R) = R \sum_{n=1}^{\infty} a_n [\alpha_s(R)/(4\pi)]^n$ . This

yields an RGE for  $R$

$$R \frac{d}{dR} m(R) = -\frac{d}{d \ln R} \delta m(R) \equiv -R \gamma_R[\alpha_s(R)],$$

$$\gamma_R[\alpha_s(R)] = \sum_{n=0}^{\infty} \gamma_n^R \left[ \frac{\alpha_s(R)}{4\pi} \right]^{n+1}. \quad (6)$$

For the kinetic scheme the RGE in  $R$  was formulated in Refs. [3, 7]. To our knowledge the full implications of Eq. (6) have not yet been studied. We will refer to  $\gamma_0^R$ ,  $\gamma_1^R$ ,  $\gamma_2^R$  as the LO, NLO, NNLO, anomalous dimensions of the RGE in  $R$ . Here  $\gamma_0^R = a_1$ ,  $\gamma_1^R = a_2 - 2\beta_0 a_1$ ,  $\gamma_2^R = a_3 - 4\beta_0 a_2 - 2\beta_1 a_1$ , so they are determined by the non-logarithmic terms in Eq. (2). For  $\mu$ -dependent renormalization schemes with a parameter  $R$ , the same RGE in Eq. (6) is obtained once we set  $\mu = R$  and define  $m(R) = m(R, R)$  and  $\delta m(R) = \delta m(R, R)$ . These schemes have consistent RG flow in the two-dimensional  $R$ - $\mu$  plane (vertically and along the diagonal). An example of such a scheme is the jet-mass,  $m^{\text{jet}}(R, \mu)$ , defined via the position-space jet-function [10]. To estimate uncertainties in the RGE in  $R$  one can set  $\mu = \kappa R$  in  $\delta m$  and determine  $\gamma_n^R(\kappa)$ , then vary about  $\kappa = 1$ .

Since the  $\mathcal{O}(\Lambda_{\text{QCD}})$  ambiguity in  $m_{\text{pole}}$  is  $R$ -independent, the derivative of  $\delta m$  in Eq. (6) ensures that  $\gamma_R[\alpha_s]$  does not contain this renormalon (a key point!). The RGE flow in the parameter  $R$  takes us from a renormalon free mass  $m(R_0)$  to the renormalon free mass  $m(R_1)$ .

Examples of LO anomalous dimensions are

$$(\gamma_0^R)^{\text{PS}} = 4C_F, \quad (\gamma_0^R)^{\text{stat}} = 2\pi C_F, \quad (\gamma_0^R)^{\text{MSR}} = 4C_F, \\ (\gamma_0^R)^{\text{kinetic}} = 16C_F/3, \quad (\gamma_0^R)^{\text{jet}} = 2e^{\gamma_E} C_F. \quad (7)$$

One can find another suitable scheme from the  $\overline{\text{MS}}$ -pole mass relation, by taking  $\overline{m}(\overline{m}) \rightarrow R$  in  $\delta m$ . We call this the MSR-scheme. All these  $\gamma_0^R$ 's are positive. Thus for large enough  $R$  we have  $d/d \ln R m(R) < 0$ , and increasing  $R$  always decreases  $m(R)$ . This sign appears as a universal feature of physical short-distance mass schemes. Now, the norm of  $\gamma_0^R$  does depend on the scheme. For a given change  $\Delta R$  it determines the amount of IR fluctuations that are added to  $m(R)$ . In a different scheme an equivalent amount of IR fluctuations can always be added to the mass with a different change  $\Delta R'$ . To see this, consider rescaling  $R = \lambda R'$  with a  $\lambda > 0$ . We demand  $\lambda \sim \mathcal{O}(1)$  to avoid large logs. Expanding  $\alpha_s(\lambda R') = \alpha_s(R') - \beta_0 \ln \lambda \alpha_s^2(R')/(2\pi) + \dots$  gives

$$\gamma_0^{R'} = \lambda \gamma_0^R, \quad \gamma_1^{R'} = \lambda [\gamma_1^R - 2\beta_0 \gamma_0^R \ln \lambda], \quad (8) \\ \gamma_2^{R'} = \lambda [\gamma_2^R - (4\beta_0 \gamma_1^R + 2\beta_1 \gamma_0^R) \ln \lambda + 4\beta_0^2 \gamma_0^R \ln^2 \lambda].$$

Thus at LO a scale change in  $R$  just modifies the norm of  $\gamma_0^R$ , and we are free to pick  $\lambda$  so that  $\gamma_0^{R'}$  is equal to the LO anomalous dimension in some other scheme. The condition  $(\gamma_0^{R'}/\gamma_0^R) \sim \mathcal{O}(1)$  identifies a class of related schemes parameterized by the scale choice for  $R$  (our  $\lambda$ ). For a top-quark in the PS, static, and MSR schemes we find  $\{\tilde{\gamma}_0^R, \tilde{\gamma}_1^R, \tilde{\gamma}_2^R\} = \{0.348, 0.108, 0.231\}^{\text{PS}}, \{0.546,$

$-0.061, 0.143\}^{\text{stat}}, \{0.348, 0.213, 0.068\}^{\text{MSR}}$ , where we let  $\tilde{\gamma}_k^R \equiv \gamma_k^R/(2\beta_0)^{k+1}$  and used Refs. [6, 12, 13]. Here and below we use 5 light running flavors.

To determine the general solution to Eq. (6) first write

$$\ln \frac{R_1}{R_0} = \int_{\alpha_0}^{\alpha_1} \frac{d\alpha_R}{\beta[\alpha_R]} = \int_{t_1}^{t_0} dt \hat{b}(t) = G(t_0) - G(t_1), \quad (9)$$

where  $\alpha_i \equiv \alpha_s(R_i)$ ,  $\alpha_R \equiv \alpha_s(R)$ ,  $t_i \equiv -2\pi/(\beta_0\alpha_i)$ , and  $t \equiv -2\pi/(\beta_0\alpha_R)$ . For the first few orders

$$\begin{aligned} \hat{b}(t) &= 1 + \frac{\hat{b}_1}{t} + \frac{\hat{b}_2}{t^2} + \frac{\hat{b}_3}{t^3} \dots, \\ G(t) &= t + \hat{b}_1 \ln(-t) - \frac{\hat{b}_2}{t} - \frac{\hat{b}_3}{2t^2} - \dots, \end{aligned} \quad (10)$$

where  $\hat{b}_1 = \beta_1/(2\beta_0^2)$ ,  $\hat{b}_2 = (\beta_1^2 - \beta_0\beta_2)/(4\beta_0^4)$ , and  $\hat{b}_3 = (\beta_1^3 - 2\beta_0\beta_1\beta_2 + \beta_0^2\beta_3)/(8\beta_0^6)$ . Note that  $G'(t) = \hat{b}(t)$ . The function  $G(t)$  allows us to define

$$\Lambda_{\text{QCD}} = R e^{G(t)} = R_0 e^{G(t_0)}, \quad (11)$$

a definition that is valid to any order in perturbation theory, and corresponds to the familiar definition of  $\Lambda_{\text{QCD}}^{(k)}$  at  $N^k\text{LL}$  order. Now by making a change of variable we obtain the all orders result

$$\begin{aligned} m(R_1) - m(R_0) &= - \int_{\ln R_0}^{\ln R_1} d \ln R \ R \gamma_R[\alpha_s(R)] \quad (12) \\ &= \Lambda_{\text{QCD}} \int_{t_1}^{t_0} dt \ \gamma_R(t) \ \frac{d}{dt} e^{-G(t)}, \end{aligned}$$

where  $\gamma_R(t) = \gamma_R[\alpha_s(R)]$ . This is a very convenient formula for the solution of the RGE in  $R$ .

Lets consider the LL solution. The RGE is

$$R \frac{d}{dR} m(R) = -R \gamma_0^R \frac{\alpha_s(R)}{4\pi}. \quad (13)$$

Here  $\gamma_R(t) = -\gamma_0^R/(2\beta_0 t)$  and with the leading terms for  $\hat{b}(t)$  and  $G(t)$ , Eq. (12) gives

$$\begin{aligned} m(R_1) - m(R_0) &= \frac{\Lambda_{\text{QCD}}^{(0)} \gamma_0^R}{2\beta_0} \int_{t_1}^{t_0} dt \ \frac{e^{-t}}{t} \quad (14) \\ &= \frac{\Lambda_{\text{QCD}}^{(0)} \gamma_0^R}{2\beta_0} [\Gamma(0, t_1) - \Gamma(0, t_0)]. \end{aligned}$$

Since  $t_1 < t_0 < 0$  the integral is convergent. Here and below we make use of the incomplete gamma function

$$\Gamma[c, t] = \int_t^\infty dx \ x^{c-1} e^{-x}, \quad (15)$$

and note that differences like the one in Eq. (14) have no contributions from the cut present in  $\Gamma[c, t]$  for  $t < 0$ . To see which perturbative terms the solution in Eq. (14) contains, recall the asymptotic expansion for  $t \rightarrow \infty$

$$\Gamma[c, t] \stackrel{\text{asym}}{=} e^{-t} t^{c-1} \sum_{n=0}^{\infty} \frac{\Gamma(1-c+n)}{(-t)^n \Gamma(1-c)}. \quad (16)$$

For  $c = 0$  this plus the LL relation  $\Lambda_{\text{QCD}}^{(0)} e^{-t} = R$  yields

$$\Lambda_{\text{QCD}}^{(0)} \Gamma[0, t] \stackrel{\text{asym}}{=} -2R \sum_{n=0}^{\infty} 2^n n! \left[ \frac{\beta_0 \alpha_s(R)}{4\pi} \right]^{n+1}. \quad (17)$$

This is a divergent series, but for Eq. (14) we have

$$\begin{aligned} m(R_1) - m(R_0) & \quad (18) \\ &= \frac{-\gamma_0^R R_1}{2\beta_0} \sum_{n=0}^{\infty} \left[ \frac{\beta_0 \alpha_1}{2\pi} \right]^{n+1} n! \left( 1 - \frac{R_0}{R_1} \sum_{k=0}^n \frac{1}{k!} \ln^k \frac{R_1}{R_0} \right) \\ &= -\frac{\gamma_0^R R_0}{2\beta_0} \sum_{n=0}^{\infty} \left[ \frac{\beta_0 \alpha_1}{2\pi} \right]^{n+1} \sum_{k=n+1}^{\infty} \frac{n!}{k!} \ln^k \frac{R_1}{R_0}, \end{aligned}$$

which is convergent since  $\beta_0 \alpha_s(R_1) \ln(R_1/R_0)/(2\pi) < 1$ . Eq. (18) displays the problem of large logs in fixed order perturbation theory for  $R_1 \gg R_0$ . The RGE in  $R$  encodes IR physics from the large order behavior of perturbation theory. It causes a rearrangement of the IR fluctuations included in the mass in going from  $R_0$  to  $R_1$  without reintroducing a renormalon.

Using Eq. (12) the solution in Eq. (14) can be extended to include all the terms up to  $N^k\text{LL}$ . Let  $[t\gamma_R(t) \hat{b}(t) e^{-G(t)} e^t (-t)^{\hat{b}_1}] \equiv -\sum_{j=0}^{\infty} S_j (-t)^{-j}$ , then

$$\begin{aligned} [m(R_1) - m(R_0)]^{N^k\text{LL}} &= \Lambda_{\text{QCD}}^{(k)} \sum_{j=0}^k S_j (-1)^j \\ &\times e^{i\pi \hat{b}_1} [\Gamma(-\hat{b}_1 - j, t_1) - \Gamma(-\hat{b}_1 - j, t_0)]. \end{aligned} \quad (19)$$

This solution is real. Here  $\Lambda_{\text{QCD}} \Gamma[c, t] \sim R \mathcal{O}(\alpha_s^{1-c})$  encodes the suppression of higher order terms, see Eq.(16). Using  $\tilde{\gamma}_k^R \equiv \gamma_k^R/(2\beta_0)^{k+1}$  the first few coefficients are

$$\begin{aligned} S_0 &= \tilde{\gamma}_0^R, \quad S_1 = \tilde{\gamma}_1^R - (\hat{b}_1 + \hat{b}_2) \tilde{\gamma}_0^R, \quad (20) \\ S_2 &= \tilde{\gamma}_2^R - (\hat{b}_1 + \hat{b}_2) \tilde{\gamma}_1^R + [(1 + \hat{b}_1) \hat{b}_2 + (\hat{b}_2^2 + \hat{b}_3)/2] \tilde{\gamma}_0^R. \end{aligned}$$

**Connection to  $m_{\text{pole}}$ .** Eq. (12) allows us to study the  $u = 1/2$  renormalon in  $m_{\text{pole}}$ . This is done by taking the limit  $R_0 \rightarrow 0$  and  $t_0 = -\ln(R_0/\Lambda_{\text{QCD}}) + \dots \rightarrow \infty$ , where we continue in the upper complex plane around the Landau pole. Since  $\lim_{R_0 \rightarrow 0} m(R_0) = m_{\text{pole}}$ , we obtain

$$m(R_1) - m_{\text{pole}} = \Lambda_{\text{QCD}} \int_{t_1}^{\infty} dt \ \gamma_R(t) \ \frac{d}{dt} e^{-B(t)}. \quad (21)$$

Upon expansion in  $\alpha_s$  the RHS reproduces  $(-\delta m)$  in Eq. (2). Eq. (21) represents an all order expression for the  $\Lambda_{\text{QCD}}$  renormalon in  $m_{\text{pole}}$ . This result can be manipulated into an equivalent expression for an inverse Borel transform of  $B(u)$ . We obtain

$$m(R) - m_{\text{pole}} = \int_0^\infty du \ B(u) \ e^{-u \frac{4\pi}{\beta_0 \alpha_s(R)}}, \quad (22)$$

$$B(u) = 2R \left[ \sum_{\ell=0}^{\infty} g_\ell Q_\ell(u) - P_{1/2} \sum_{\ell=0}^{\infty} g_\ell \frac{\Gamma(1 + \hat{b}_1 - \ell)}{(1 - 2u)^{1 + \hat{b}_1 - \ell}} \right],$$

$$P_{1/2} = \sum_{k=0}^{\infty} \frac{S_k}{\Gamma(1 + \hat{b}_1 + k)}, \quad (23)$$

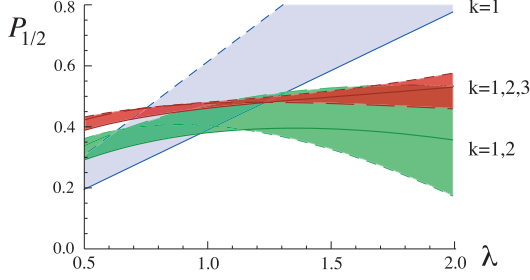


FIG. 1: Convergence of the sum-rule for  $P_{1/2}$  for  $m_{\text{pole}}$ .

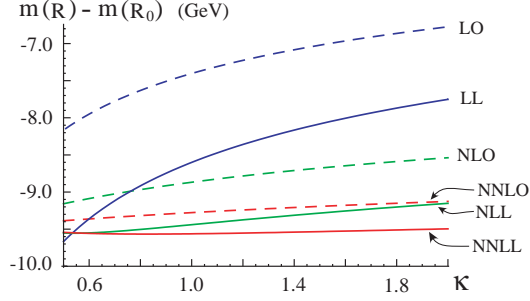


FIG. 2: Top-mass scheme conversion from  $R_0 = 3 \text{ GeV}$  to  $R = 163 \text{ GeV}$ . Shown are fixed order results (LO, NLO, NNLO) and RGE results (LL, NLL, NNLL), both in the MSR scheme.

$$Q_\ell(u) = \sum_{k=0}^{\infty} \frac{S_k(2u)^{k+\ell} {}_2F_1(1, 1+\hat{b}_1+k, 2+\hat{b}_1-\ell, 1-2u)}{(1+\hat{b}_1-\ell) \Gamma(k+\ell)}.$$

Here  $e^{G(t)} e^{-t} (-t)^{-\hat{b}_1} \equiv \sum_{\ell=0}^{\infty} g_\ell (-t)^{-\ell}$ , so  $g_0 = 1$ ,  $g_1 = \hat{b}_2$ ,  $g_2 = (\hat{b}_2^2 - \hat{b}_3)/2$ , etc. The normalization  $P_{1/2}$  multiplies all terms singular at  $u = 1/2$  in Eq. (22). Since  $\gamma_R(t)$  is free of the  $u = 1/2$  renormalon, the large order behavior of  $\gamma_k^R$  is dominated by the next pole at  $u = \rho > 1/2$ . This implies that asymptotically for large  $k$ ,  $\gamma_k^R \sim k! (\beta_0)^{k+1} \rho^{-k}$ . Given that the sum for  $\beta[\alpha_s]$  (and hence  $\sum_\ell g_\ell$ ) converges,  $S_k \sim \tilde{\gamma}_k \sim k! (2\rho)^{-k}$ , so the sum over  $k$  in  $P_{1/2}$  converges. Since  $Q_\ell(1/2) = \sum_{k=0}^{\infty} S_k [(1+\hat{b}_1-\ell)\Gamma(k+\ell)]^{-1}$ , all sums over  $k$  are *absolute convergent* for  $u$  close to  $1/2$ . From Eq. (22) the large- $n$  asymptotic behavior for any  $m(R) - m_{\text{pole}}$  is  $a_{n+1} \sim a_{n+1}^{\text{asym}} \equiv P_{1/2} (2\beta_0)^{n+1} \sum_{\ell=0}^{\infty} g_\ell \Gamma(1+\hat{b}_1-\ell+n)$ , and this series in  $\ell$  agrees precisely with the behavior expected from the  $\Lambda_{\text{QCD}}$  ambiguity [8]. Thus Eq. (23) gives a convergent sum-rule for the normalization  $P_{1/2}$ .

$P_{1/2}$  allows us to test for a  $u = 1/2$  renormalon without relying on the  $n_f$ -bubble chain. Any Borel summable series of  $a_n$ 's in Eq. (2) leads to a  $P_{1/2}$  that rapidly goes to zero. The largest physical series of  $a_n$ 's that sums to  $P_{1/2} = 0$  has a  $u = 1$  renormalon, whereas  $P_{1/2} \neq 0$  for any  $u = 1/2$  pole. Due to the universality of the  $u = 1/2$

renormalon of  $m_{\text{pole}}$ , its  $P_{1/2}$  is a unique scheme independent number. In Fig. 1 we plot the sum of terms for this  $P_{1/2}$  up to  $k = 0$  (light/blue),  $k = 1$  (medium/green), and  $k = 2$  (dark/red). We show the PS (solid), static (dashed), and MSR-schemes (long-dashed), which are each generalized to a class of schemes with  $\lambda \in [1/2, 2]$  using Eq. (8). The convergence is clearly visible, and we estimate  $P_{1/2} = 0.47 \pm 0.10$ . For comparison, the widely used light-fermion bubble chain [8] (large- $n_f$  with naive-non-Abelianization,  $n_f \rightarrow -3\beta_0/2$ ), gives an overestimate,  $P_{1/2} = 0.80$ . A different series for  $P_{1/2}$  was derived in Refs. [14, 15], evaluating  $(1-2u)^{1+\hat{b}_1} B(u)$  in an expansion about  $u = 0$ , at  $u = 1/2$ . It gives  $P_{1/2} \simeq 0.48$ , in agreement with our result. One can use  $a_n^{\text{asym}}$  in Eq. (2) to define a mass-scheme and study its  $R$  dependence [15], which however suffers from the uncertainty in  $P_{1/2}$ .

Top-quark mass measurements from jets rely on an underlying Breit-Wigner, and should be considered as values  $m(R_0)$  in some scheme with  $R_0 \sim \Gamma_t$  [11]. The top  $\overline{\text{MS}}$  scheme has  $R \simeq 163 \text{ GeV} \gg R_0$  so a fixed order conversion to  $\overline{\text{MS}}$  involves large logs. If we measure the MSR mass at  $R_0$  and run to  $R = [\overline{m}(\overline{m})]^{\overline{\text{MS}}}$  then we directly get this  $\overline{\text{MS}}$  mass. In Fig. 2 we compare conversions between MSR-schemes with  $R_0 = 3 \text{ GeV}$  and  $R = 163 \text{ GeV}$ , using a fixed order expansion in  $\alpha_s(\mu) = \alpha_s(\kappa R)$  (dashed curves), and the solution of the RGE in Eq. (19) for  $\gamma_i^R$  obtained with  $\mu = \kappa R$  (solid curves). Varying  $\kappa$  gives a measure for the residual uncertainty at a given order. The plot shows that Eq.(19) converges rapidly, with flat  $\kappa$  dependence at NNLL. Also the RGE results display better convergence than the standard fixed order expressions. Comparisons using the PS and static schemes yield the same conclusion, with similar convergence. Taking the Tevatron mass  $172.6 \pm 1.4 \text{ GeV}$  [16] as  $m^{\text{MSR}}(R_0)$ , we obtain  $\overline{m}(\overline{m}) = 163.0 \pm 1.3_{-3}^{+6} \pm .05 \text{ GeV}$ . The first error is experimental, the second takes  $R_0 = 3_{-2}^{+6} \text{ GeV}$  to account for the scheme uncertainty, and the third is the uncertainty in our NNLL conversion. This assumes the experimental error accounts for hadronic uncertainties.

The infrared RG analysis performed here can be generalized to study higher order renormalons and quantities other than quark masses. For an infrared sensitive matrix element of  $\mathcal{O}(\Lambda_{\text{QCD}}^N)$  the anomalous dimension will have terms  $R^N \alpha_s^n(R)$ , and the corresponding sum-rule will provide info on the Borel singularity at  $u = N/2$ .

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